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TITLE: RELATIVE SYMMETRIES OF DIFFERENTIAL EQUATIONS

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# RELATIVE SYMMETRIES OF DIFFERENTIAL EQUATIONS

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Let $\Delta : J^\infty\nu \to J^\infty\pi$ be a differential operator, where $J^\infty\nu$ (resp. $J^\infty\pi$) is the infinite-jet bundle of the bundle $\nu : F \to M$ (resp. $\pi : E \to M$). Let $I^1_\nu$ be the Cartan submodule of the module $\Lambda^1(K_\nu)$ of 1-forms over the ring $K_\nu = C^\infty(J^\infty\nu)$. Among all derivations of $K_\pi$ into $K_\nu$ along $\Delta^*$, we classify those which map $I^1_\pi$ into $I^1_\nu$. They turn out to be quasi-evolution equations.

## 1. INTRODUCTION

Let $\pi : E \to M$, $\nu : F \to M$ be bundles (smooth, like everything else in the paper). Let $\pi_k : J^k\pi \to M$, $\pi_{k,\ell} : J^k\pi \to J^\ell\pi$ be the corresponding jet bundles, denote $J^\infty\pi = \lim\text{proj}\, J^k\pi$, $K_\pi = C^\infty(J^\infty\pi) = \lim\text{ind}\, C^\infty(J^k\pi)$. Let $\bar\Delta : J^s\nu \to J^0\pi = E$ be a bundle map (over M), which can be thought of as a differential operator $\tilde\Delta : \Gamma(\nu) \to \Gamma(\pi)$, where $\Gamma(\nu)$ denotes the sheaf of sections of the bundle $\nu$ : $\tilde\Delta(\gamma) = \bar\Delta\cdot(j_s(\nu)(\gamma))$, $\forall\gamma\in\Gamma(\nu)$, where $j_s = j_s(\nu) : \Gamma(\nu) \to \Gamma(\nu_s)$ denotes the natural lift. Tangent planes to graphs $\{j_k(\gamma)(M) \mid \gamma\in\Gamma(\nu)\}$ form the Cartan distribution in $J^k\nu$. Its annihilator in $\Lambda^1(J^k\nu)$ is the k-th Cartan submodule $I_k(\nu)$. The Cartan submodule $I^1(\nu)$ in $\Lambda^1(\nu) = \Lambda^1(J^\infty\nu) = \lim\text{ind}\,\Lambda^1(J^k\pi)$ is defined by the formula $I^1(\nu) = \lim\text{ind}\, I_k(\nu)$. Let us denote by $\Delta$ the natural lift of $\bar\Delta$ into $J^\infty\nu$, $\Delta : J^\infty\nu \to J^\infty\pi$. Then $\Delta^*(I^1_\pi) \subset I^1_\nu$ (lemma II 2.14 [3]).

We consider the following problem: find the set $\mathfrak{D}^{qev}(\Delta)$ of all derivations $Z : K_\pi \to K_\nu$ along the homomorphism $\Delta^*$, which map $I^1_\pi$ into $I^1_\nu$. There are at least three motivations for this problem:

A. In the case $\pi = \nu$, $\Delta = \text{id}$, the set of all such Z's is the set of evolution derivations $\mathfrak{D}^{ev}(\pi)$; in local coordinates, the equations of trajectories of these evolution derivations are evolution equations (Proposition 1 [2]; Theorem I 5.6 [3]). (In the engineering literature, these derivations pass under the misleading name "Lie-Bäcklund transformations".)

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B. Such Z's arise in practice as the "generalized sine-Gordon equations" associated with classical simple complex Lie algebras ([4],[6]) and even with Kac-Moody Lie algebras ([1]).
C. Let $U \subset J^k\pi$ be a closed set considered as a differential equation: $\gamma\varepsilon\Gamma(\pi)$ is a solution if $(j_k(\gamma))(M) \subset U$. Let $\bar{U} \subset J^\infty\pi$ be the infinite prolongation of U. Then the symmetries of $\bar{U}$ are those evolution derivations $X\varepsilon D^{ev}(\pi)$ which preserve the ideal $\mathcal{F}(\bar{U})$ of functions from $K_\pi$ vanishing on $\bar{U}$. Suppose, however, that $\bar{V} \subset J^\infty\nu$ is another equation and $\Delta(\bar{V}) \subset \bar{U}$. Then more general symmetries of $\bar{U}$ will be those Z's which map $\mathcal{F}(\bar{U})$ into $\mathcal{F}(\bar{V})$. That such relative symmetries are useful was demonstrated in a spectacular tour-de-force by Vinogradov and Krasil'shchik who used nonlocal symmetries to compute all (absolute) symmetries of the Korteweg-de Vries equation ([5]).

## 2. CLASSIFICATION

Denote by $\mathfrak{D}(\pi_\infty)$ the $K_\pi$-module of derivations of $C^\infty(M)$ into $K_\pi$ along $\pi_\infty^*$, where $\pi_\infty : J^\infty\pi \to M$ is the natural projection. Note that $\mathfrak{D}(\pi_\infty)$ is generated over $K_\pi$ by the Lie algebra $\mathfrak{D}(M)$ of vector fields on M. If $X \varepsilon \mathfrak{D}(\pi_\infty)$ then its lift $\bar{X} = \bar{X}_\pi \varepsilon \mathfrak{D}(K_\pi)$ into the Lie algebra of derivations of $K_\pi$ is uniquely defined by the universal property $j_\infty(\gamma)^*\bar{X} = j_\ell(\gamma)^* Xj_\infty(\gamma)^*$, $\forall\gamma\varepsilon\Gamma(\pi)$, where $\ell$ is such that $X(C^\infty(M)) \subset C^\infty(J^\ell\pi)$. The set of all such $\bar{X}$'s is denoted by $\overline{\mathfrak{D}(\pi_\infty)}$ and is a Lie algebra and a $K_\pi$-module (Theorem I 3.6 [3]). The annihilator of $\overline{\mathfrak{D}(\pi_\infty)}$ in $\Lambda^1(K_\pi)$ is nothing but the Cartan submodule $I^1_\pi$. [This is the definition of the Cartan submodule; the fact that the corresponding distribution is spanned by the tangent planes of graphs of jets of sections of $\pi$ is a corollary (Theorem I 4.4 [3]).]

If $X \in \mathfrak{D}(M)$ then the lifts $\bar{X}_\nu$ and $\bar{X}_\pi$ are $\Delta$-related: $\bar{X}_\nu\Delta^* = \Delta^*\bar{X}_\pi$ (Lemma II 2.13 [3].) Obviously, if $X \in \mathfrak{D}(\pi_\infty)$, then again there exists a unique $\bar{X}_\nu \varepsilon \overline{\mathfrak{D}(\nu_\infty)}$ such that $\bar{X}_\nu\Delta^* = \Delta^*\bar{X}_\pi$; the resulting map $\overline{\mathfrak{D}(\pi_\infty)} \to \overline{\mathfrak{D}(\nu_\infty)}$ is a Lie algebra homomorphism.

<u>Lemma 2.1</u>. Let $\phi : K_1 \to K_2$ be a homomorphism of commutative rings $K_1$ and $K_2$, let $X_1 \in \mathfrak{D}(K_1)$ and $X_2 \in \mathfrak{D}(K_2)$ be two $\phi$-related derivations. Let $\mathfrak{D}(\phi)$ be a $K_2$-module of derivations of $K_1$ into $K_2$ along $\phi$. Then for any $Z \in \mathfrak{D}(\phi)$, $(X_2Z - ZX_1) \varepsilon \mathfrak{D}(\phi)$.

<u>Proof</u>. Obvious.

Recall that if $\omega\varepsilon\Lambda^1(K)$, $X,Z \varepsilon \mathfrak{D}(K)$, then the Lie derivative of $\omega$ with respect to Z is defined by the formula $[Z(\omega)](X) = Z(\omega(X))-\omega([Z,X])$.

<u>Lemma 2.2</u>. In the notations of lemma 2.1, $\mathfrak{D}(\phi)$ acts by derivations along $\phi$ on $\Lambda^1(K_1)$ with values in $\Lambda^1(K_2)$. In particular, for $\omega\varepsilon\Lambda^1(K_1)$

$$[Z(\omega)](X_2) = Z(\omega(X_1)) - \omega(ZX_1-X_2Z) , \qquad (2.3)$$

where on the right hand side the pairing between $\Lambda^1(K_1)$ and $\mathfrak{D}(\phi)$ is understood naturally : $(fdg)(Z) = \phi(f)Z(g)$ , $\forall f,g\varepsilon K_1$.

Again, the proof is obvious.

Now we can handle the problem of classification of elements of $\mathfrak{D}^{qev}(\Delta)$. Let $Z\in\mathfrak{D}^{qev}(\Delta)$, that is, $Z(I^1_\pi)\subset I^1_\nu$. Take any $\omega\in I^1_\pi = \mathrm{Ann}(\overline{\mathfrak{D}(\pi_\infty)})$. Then $Z(\omega)\in I^1_\nu = \mathrm{Ann}(\overline{\mathfrak{D}(\nu_\infty)}) = \mathrm{Ann}(\overline{\mathfrak{D}(M)}_\nu)$ iff, $\forall X\in\mathfrak{D}(M)$, $[Z(\omega)](\bar{X}_\nu) = 0$. By formula (2.3), this is equivalent to $0 = Z(\omega(\bar{X}_\pi)) - \omega(Z\bar{X}_\pi - \bar{X}_\nu Z)$. But $\omega(\bar{X}_\pi) = 0$ since $\omega\in I^1_\pi$. Thus $(Z\bar{X}_\pi - \bar{X}_\nu Z)$ must belong to the kernel of $I^1_\pi$, that is, we must have

$$(Z\bar{X}_\pi - \bar{X}_\nu Z)\ \in\ K_\nu\Delta^*\overline{\mathfrak{D}(M)}_\pi,\ \forall X\in\mathfrak{D}(M). \tag{2.4}$$

Theorem 2.5. Every $Z\in\mathfrak{D}^{qev}(\Delta)$ is uniquely defined by its value $Z\cdot\pi^*_{\infty,o}$. Conversely, any derivation $\tilde{Z}\in\mathfrak{D}(\pi_{\infty,o}\Delta)$ is uniquely lifted in $\mathfrak{D}(\phi)$ to become $Z\in\mathfrak{D}^{qev}(\Delta)$, such that $Z\cdot\pi^*_{\infty,o} = \tilde{Z}$.

Proof. To study (2.4), first notice that, like in the absolute case $(\pi = \nu,\ \Delta = \mathrm{id})$, one has a direct sum decomposition

$$\mathfrak{D}(\Delta) = \overline{\mathfrak{D}(\nu_\infty)}\cdot\Delta^* \oplus \mathfrak{D}(\Delta)^{vert}\ , \tag{2.5}$$

where $\mathfrak{D}(\Delta)^{vert} := \{Z\in\mathfrak{D}(\rho)\,|\,Z\cdot\pi^*_\infty = 0\}$, and decomposition (2.6) is provided by the formula $Z = \overline{(Z\cdot\pi^*_\infty)}_\nu\circ\Delta^* + [Z - \overline{(Z\cdot\pi^*_\infty)}_\nu\circ\Delta^*]$. Since $Z\cdot\pi^*_\infty\in\mathfrak{D}(\nu_\infty) = \mathfrak{D}(\Delta)|_{C^\infty(M)}$, then $Z_1 := \overline{(Z\cdot\pi^*_\infty)}_\nu\in\mathfrak{D}(\nu_\infty)$ and (2.4) for $Z = Z_1\Delta^*$ is obviously satisfied. Therefore we shall restrict ourselves to vertical $Z$'s $\in\mathfrak{D}(\Delta)^{vert}$ only.

Let $(x_1,\dots,x_m)$ be local coordinates in $M$, $\{q^a_\sigma\,|\,a = 1,\dots \dim E - \dim M,\ \sigma\in\mathbb{Z}^m_+\}$ be standard local coordinates on $J^\infty\pi$, and $\{p^b_\sigma\,|\,b = 1,\dots,\dim F - \dim M,\ \sigma\in\mathbb{Z}^m_+\}$ be local coordinates on $J^\infty\nu$. Let, locally, $Z = \sum A^a_\sigma\Delta^*\dfrac{\partial}{\partial q^a_\sigma}$, $A^a_\sigma\in K_\nu$. It is enough to check (2.4) for the basis vector fields $X = \dfrac{\partial}{\partial x_i}\in\mathfrak{D}(M)$. Since $\overline{\left(\dfrac{\partial}{\partial x_i}\right)}_\pi = \dfrac{\partial}{\partial x_i} + q^q_{\sigma+i}\dfrac{\partial}{\partial q^a_\sigma}$ (using summation over repeated indices), we have

$$Z\bar{X}_\pi - \bar{X}_\nu Z = \left(A^a_\sigma\Delta^*\frac{\partial}{\partial q^a_\sigma}\right)\left(\frac{\partial}{\partial x_i} + q^b_{\mu+i}\frac{\partial}{\partial q^b_\mu}\right) -$$

$$- \left(\frac{\partial}{\partial x_i} + p^b_{\mu+i}\frac{\partial}{\partial p^b_\mu}\right)\left(A^a_\sigma\Delta^*\frac{\partial}{\partial q^a_\sigma}\right) = \left[\text{since } \Delta^*\overline{\left(\frac{\partial}{\partial x_i}\right)}_\pi = \overline{\left(\frac{\partial}{\partial x_i}\right)}_\nu\right] =$$

$$= \left\{-\left[\overline{\left(\frac{\partial}{\partial x_i}\right)}_\nu (A^a_\sigma)\right]\cdot\Delta^*\frac{\partial}{\partial q^a_\sigma} + A^a_\sigma\Delta^*\left[\frac{\partial}{\partial q^a_\sigma},\ \overline{\left(\frac{\partial}{\partial x_i}\right)}_\pi\right]\right\} =$$

$$= \{[-(\overline{\frac{\partial}{\partial x_i}})_\nu (A^a_\sigma) + A^a_{\sigma+i}]\Delta^* \frac{\partial}{\partial q^a_\sigma}\} .$$

This last expression must belong to $K_\nu \Delta^* \overline{\mathfrak{D}(M)}_\pi$. Since there are no components along M, it must vanish, and this happens iff $A^a_{\sigma+i} = (D_i)_\nu (A^a_\sigma)$, where $(D_i)_\nu$ stands for $\overline{(\partial/\partial x_i)}_\nu$. Thus, $A^a_\sigma = (D^\sigma)_\nu(A^a)$, $(D^\sigma)_\nu := (D_{i_1})_\nu^{\sigma_1} \cdots (D_{i_m})_\nu^{\sigma_m}$, and $A^a$'s are arbitrary.

## 3. TRAJECTORIES

Ordinary differential equations are equations of trajectories of vector fields on manifolds. Analogously, evolution equations are equations of trajectories of vertical evolution derivations (Theorem I 5.6 [3]). (The reason for considering only vertical fields is explained in §I 5.3 [3]; for nonvertical fields, equations become overdetermined.) Now let $Z \in \mathfrak{D}^{qev}(\Delta)$, and consider Z to be vertical. A trajectory of Z is a one-parameter (t) family of sections $\gamma = \gamma(t): M \to F$ such that $[j(\nu)(\gamma)]^* \circ Z = \frac{\partial}{\partial t} \circ [j(\pi)(\Delta\gamma)]^*$. Let us find a coordinate version of the last equation. Let locally $Z = (D^\sigma)_\nu (A^a) \cdot \Delta^* \, \partial/\partial q^a_\sigma$. Then $0 = [j(\nu)(\gamma)]^* \circ Z - \frac{\partial}{\partial t} \circ [j(\pi)(\Delta\gamma)]^* =$

$$= [j(\nu)(\gamma)]^* \{[(D^\sigma)_\nu (A^a)]\Delta^* \frac{\partial}{\partial q^a_\sigma}\} - (\frac{\partial}{\partial t}[(q^a_\sigma)^*(\Delta\gamma)] \cdot [j(\pi)(\Delta\gamma)]^* \frac{\partial}{\partial q^a_\sigma} =$$

$$= D^\sigma([j(\nu)(\gamma)]^*(A^a)) \cdot [j(\pi)(\Delta\gamma)]^* \frac{\partial}{\partial q^a_\sigma} - \{\frac{\partial}{\partial t} D^\sigma([j(\pi)(\Delta\gamma)]^*(q^a))\} \cdot [j(\pi)(\Delta\gamma)]^* \frac{\partial}{\partial q^a_\sigma}$$

where $D^\sigma := \overline{(\partial/\partial x_{i_1})}^{\sigma_1} \cdots \overline{(\partial/\partial x_{i_m})}^{\sigma_m}$. Since $[\partial/\partial t, D^\sigma] = 0$, the above equality is reduced to

$$\frac{\partial}{\partial t} \{[j(\pi)(\Delta\gamma)]^* (q^a)\} = [j(\nu)(\gamma)]^* (A^a) . \tag{3.1}$$

Thus we obtain the coordinate form of quasievolution equations.

Remark 3.2. In contrast to the evolution equations, quasievolution ones need not be formally integrable. Obviously, integrability of a generic Z depends only upon Δ. I conjecture that this integrability depends only upon dimensions and codimensions of the finite number of prolongations of the map $\bar{\Delta} : J^a\nu \to E$.

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